Answer all questions
Open book, notes, and handouts

Name: SOLUTION

1. (25 points)
The wall section shown in Figure 1 consists of 4 in. face brick (material A) on the outside and 6 in. concrete (material B) on the inside. The inside convection coefficient is 1.64 Btu/hr ft^2°F and the outside convection coefficient is 5.88 Btu/hr ft^2°F. The thermal properties of the materials are given in Table 1. An explicit finite difference scheme is to be employed to analyze the one-dimensional heat transfer through the wall. Spatial nodes are to be placed at the two surfaces and at one-inch intervals through the wall.

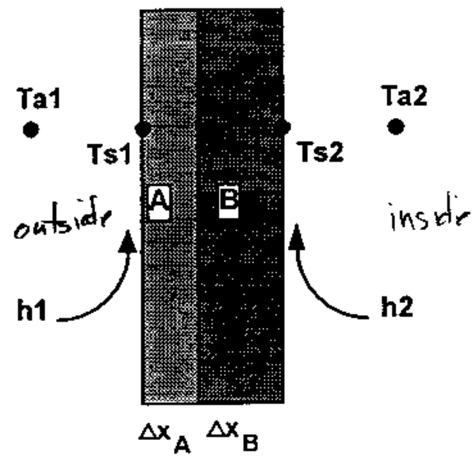


Figure 1: Sample wall section

a) List the complete set of criteria that must be considered to determine the maximum time-step for stability.

b) What is the maximum time-step?

Table 1: Properties of wall materials

Property	Material A	Material B
Density	130 lb/ft^3	140 lb/ft^3
Conductivity	0.75 Btu/hr ft°F	1.00 Btu/hr ft°F
Specific Heat	0.19 Btu/lb°F	0.22 Btu/lb°F
Thermal Diffusivity	8.43 x 10^{-6} ft^2/s	9.02 x 10^{-6} ft^2/s

For stability:

interior $1 - 2 F_o > 0$ $\qquad F_o = \frac{\alpha \Delta t}{\Delta x^2}$

exterior $1 - 2 F_o (1 + Bi) > 0$ $\qquad Bi = \frac{h \Delta x}{k}$

interior: $\Delta t < \frac{\Delta x^2}{2\alpha}$ $\qquad \Delta x = \frac{1}{12}$ ft

exterior $\Delta t < \frac{\Delta x^2}{2\alpha(1 + h\Delta x / k)}$

Material A:

interior: $\Delta t < \frac{1}{2(144)(8.43 \times 10^{-6})}$

$\Delta t < 412$ sec

exterior $Bi = \frac{5.88}{12(0.75)} = 0.653$

$\Delta t < \frac{412}{1.653}$ $\boxed{\Delta t < 249 \text{ s}}$

Material B:

interior $\Delta t < \frac{1}{2(144)(9.02 \times 10^{-6})}$

$\Delta t < 385$ sec

exterior $Bi = \frac{1.64}{12(1.00)} = 0.137$

$\Delta t < \frac{385}{1.137}$ $\quad \Delta t < 339$ s

2. (20 points)
Consider a linear and homogeneous thermal system initially at zero. The response of the system, *u(t)*, to a unit step input can be characterized by the unit response *U(t)*.

$$U(t) = 1 - e^{-t/b}$$

Instead of the unit step input, let the system be exposed to a sinusoidal input of the form

$$F(t) = \sin(t) \qquad \frac{dF(t)}{dt} = \cos t$$

Using Duhamel's theorem and the attached table of integrals, develop an equation for the response of the sytem.

$$u(t) = \int_0^t U(t-\tau)\frac{dF(\tau)}{d\tau}\,d\tau$$

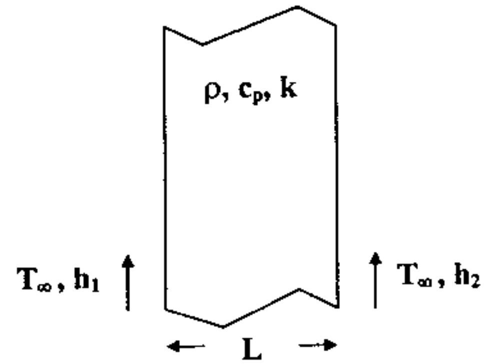
$$u(t) = \int_0^t \left(1 - e^{-(t-\tau)/b}\right)\cos\tau\,d\tau$$

$$= \left[\sin\tau\right]_0^t - e^{-t/b}\left[\frac{e^{\tau/b}}{1/b^2+1}\left(\frac{1}{b}\cos\tau + \sin\tau\right)\right]_0^t$$

$$= \sin t - \frac{1}{1/b^2+1}\left(\frac{1}{b}\cos t + \sin t\right) + \frac{e^{-t/b}}{1/b^2+1}\left(\frac{1}{b}\right)$$

$$= \frac{1}{b^2+1}\left[\sin t + b\left(e^{-t/b} - \cos t\right)\right]$$

3. (10 points)
The wall shown in the figure has been characterized using transfer function coefficients for use in a building computer simulation program. The transfer function coefficients have been generated for a specific set of wall construction and thermal characteristics and convection heat transfer coefficients using one hour time intervals.



a) Briefly describe the assumptions and limitations in using these transfer function coefficients.

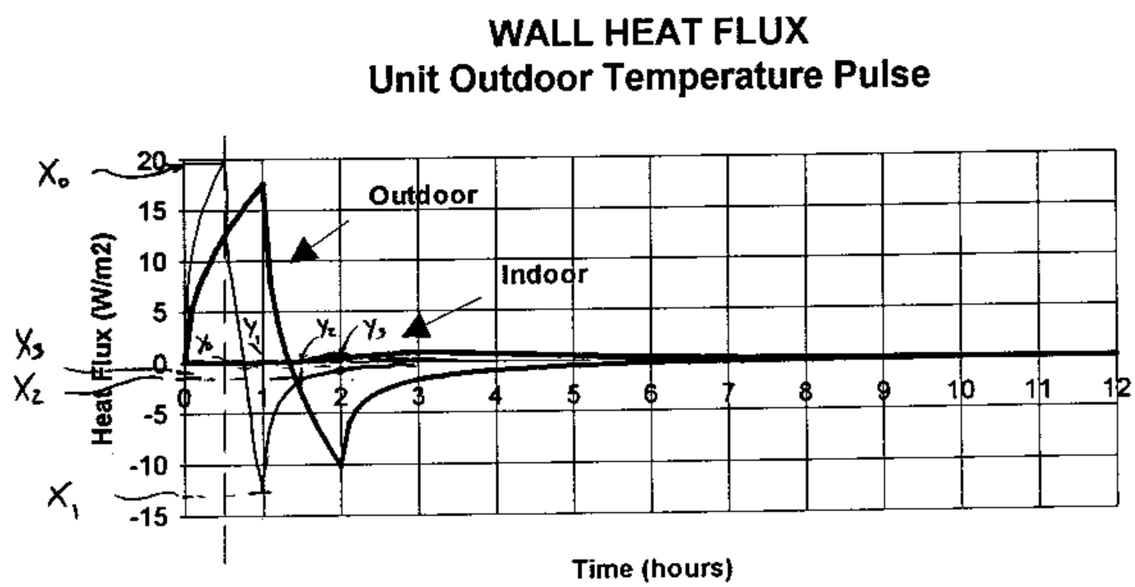
b) Briefly describe how to include the effects of solar radiation on the outside surface of the wall using the transfer function approach.

a) $\Delta t, L, \rho, c_p, k, h_1, h_2$ constant
one dimensional

b) Replace T_{air} with T_{solair}

$$T_{solair} = T_{air} + \frac{\alpha I}{h}$$

4. (20 points)
Building thermal analyses are often performed on an hourly basis using measured weather data. Response factors are typically developed for one hour time steps using unit triangular pulses with a base of 2 hours. Figure 2 shows the heat flux at the indoor and outdoor surface for a wall exposed to a unit triangular pulse in outdoor temperature with a base of 2 hours. Instead, consider the case of a response factor analysis with half hour time steps. Qualitatively sketch (on the figure) the expected heat flows to a unit triangular pulse in outdoor temperature with a base of 1 hour, and identify the X_j and Y_j response factors for the analysis with half hour time steps.



Main Features · Outdoor — higher peak
lower valley
quicker to zero

Indoor — lower peak
quicker to zero

5. (25 points)
A typical frame wall with 4 inches insulation has the following transfer function coefficients for use with 1 hour time steps. (b_n and c_n have units of W/m^2°C)

n	b_n	d_n
0	0.00270	1.00000
1	0.05585	-0.81542
2	0.06706	0.20105
3	0.00944	-0.01425

$$\sum_{n=0}^{\infty} c_n = 0.13505$$

The wall is initially at 20°C. Assume no solar radiation on the outside surface. Complete the following table.

Time (hr)	Outdoor Temperature (°C)	Indoor Temperature (°C)	Indoor Heat Flux (W/m^2)
0	20.0	20.0	0.0
1	22.5	20.0	0.00675
2	24.2	20.0	0.15647
3	25.8	20.0	0.54411

$$q(1) = 0.0027(22.5) + (0.05585 + 0.06706 + 0.00944)(20) - 0.13505(20) - 0$$

$$q(1) = 0.00675 \text{ W/m}^2$$

$$q(2) = 0.0027(24.2) + 0.05585(22.5) + (0.06706 + .00944)(20) - 0.13505(20) - (-0.81542)(0.00675)$$

$$q(2) = 0.15647 \text{ W/m}^2$$

$$q(3) = 0.0027(25.8) + 0.05585(24.2) + 0.06706(22.5) + 0.00944(20) - 0.13505(20) - (-0.81542)(0.15647) - 0.20105(0.00675)$$

$$q(3) = 0.54411 \text{ W/m}^2$$

567.1. $\int xe^{ax}dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right].$

567.2. $\int x^2e^{ax}dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right].$

567.3. $\int x^3e^{ax}dx = e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right].$

567.8. $\int x^ne^{ax}dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}dx.$

567.9. $$\int x^ne^{ax}dx = e^{ax}\left[\frac{x^n}{a} - \frac{nx^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} - \cdots + (-1)^{n-1}\frac{n!x}{a^n} + (-1)^n\frac{n!}{a^{n+1}}\right], \qquad [n \geqq 0].$$

568.1. $$\int \frac{e^{ax}dx}{x} = \log|x| + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \cdots + \frac{a^nx^n}{n\cdot n!} + \cdots, \qquad [x^2 < \infty].$$

568.11. For $\int \frac{c^xdx}{x}$, note that $c^x = e^{x\log c}$.

568.2. $\int \frac{e^{ax}dx}{x^2} = -\frac{e^{ax}}{x} + a\int\frac{e^{ax}dx}{x}.$ [See **568.1.**]

568.3. $\int \frac{e^{ax}dx}{x^3} = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2}\int\frac{e^{ax}dx}{x}.$ [See **568.1.**]

568.8. $\int \frac{e^{ax}dx}{x^n} = -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int\frac{e^{ax}dx}{x^{n-1}},$ $[n > 1]$.

568.9. $$\int \frac{e^{ax}dx}{x^n} = -\frac{e^{ax}}{(n-1)x^{n-1}} - \frac{ae^{ax}}{(n-1)(n-2)x^{n-2}} - \cdots - \frac{a^{n-2}e^{ax}}{(n-1)!x} + \frac{a^{n-1}}{(n-1)!}\int\frac{e^{ax}dx}{x},$$ $[n > 1]$. [See **568.1.**]

569. $\int \frac{dx}{1+e^x} = x - \log(1+e^x) = \log\frac{e^x}{1+e^x}.$

569.1. $\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\log|a+be^{px}|.$

570. $\int \frac{xe^xdx}{(1+x)^2} = \frac{e^x}{1+x}.$ 570.1. $\int \frac{xe^{ax}dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$

575.1. $\int e^{ax}\sin x\,dx = \frac{e^{ax}}{a^2+1}(a\sin x - \cos x).$

575.2. $\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right).$

575.3. $$\int e^{ax}\sin^3 x\,dx = \frac{e^{ax}}{a^2+9}\left[a\sin^3 x - 3\sin^2 x\cos x + \frac{6(a\sin x - \cos x)}{a^2+1}\right].$$

575.9. $$\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx.$$

576.1. $\int e^{ax}\cos x\,dx = \frac{e^{ax}}{a^2+1}(a\cos x + \sin x).$

576.2. $\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\cos^2 x + 2\sin x\cos x + \frac{2}{a}\right).$

576.3. $$\int e^{ax}\cos^3 x\,dx = \frac{e^{ax}}{a^2+9}\left[a\cos^3 x + 3\sin x\cos^2 x + \frac{6(a\cos x + \sin x)}{a^2+1}\right].$$

576.9. $$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x}{a^2+n^2}(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx.$$ [Ref. 2, p. 141.]

577.1. $\int e^{ax}\sin nx\,dx = \frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx).$

577.2. $\int e^{ax}\cos nx\,dx = \frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx).$

[Ref. 7, p. 9.]

MIDTERM TAKE-HOME EXAM

SOLUTION

In Homework #3 you developed an analytical solution for the heat fluxes at the surfaces of a plane wall exposed to a triangular pulse in surface temperature. In Homework #4/5, you used this solution with superposition theory to develop the solution for the time-varying outside surface temperature profile shown in Figure 1. The solution was based on the principle that the continuous temperature profile could be approximated as the sum of a series of triangular profiles, giving a trapezoidal approximation to the function. For this exam, you will essentially repeat these developments using a *rectangular* temperature pulse to represent the continuous variation in outdoor temperature.

Consider a wall, 20 cm thick, constructed of common brick. The thermal properties of the wall are given in Table 1. The entire wall is initially at 20°C and inside surface temperature is held constant at 20°C.

a) Using superposition and the response of a plane wall to a step change in surface temperature, develop general expressions for the heat flux at both surfaces as a function of time for a rectangular pulse in outside surface temperature. Consider the wall to initially be at a uniform temperature of T_o, and the surface triangular temperature to reach a maximim temperature of T_p and d as shown in Figure 2.

Define X_j,Y_j, and Z_j response factors in the same way as previously defined, except that a rectangular pulse of base d is used in place of a triangular pulse of base $2d$. In other words, the X_j response factor describes the heat transfer from the outdoors into the outside surface at time $t=jd$ for a unit rectangular pulse in outdoor temperature, centered around $t=0$ with base d. The Y_j response factor describes the heat transfer into the room through the inside surface for a unit triangular pulse in outdoor temperature. (The Y_j response factor also describes the heat transfer out of the outside surface to the outdoors for a unit triangular pulse in indoor temperature.) The Z_j response factor describes the heat transfer from the room to the inside surface for a unit triangular pulse in indoor temperature.

b) For $d = 1$ hour, determine the values of X_j,Y_j, and Z_j.

Consider that the outside surface temperature varies with time according to the steady periodic profile of Figure 1. (That is, the outside surface has been exposed to this periodic profile for a long period of time and will continue to be exposed to the profile in the future.)

c) Develop the steady periodic heat flux profiles at the inside and outside surfaces using Duhamel's theorem.
d) Develop the steady periodic heat flux profiles at the inside and outside surfaces using the response factor method.
e) Compare the solutions of c), and d) and discuss any differences.

Table 1: Properties of common brick

Property	Value
Density	1600 kg/m^3
Conductivity	0.69 W/m C
Specific Heat	0.84 kJ/kg C
Thermal Diffusivity	5.2 x 10^{-7} m^2/s

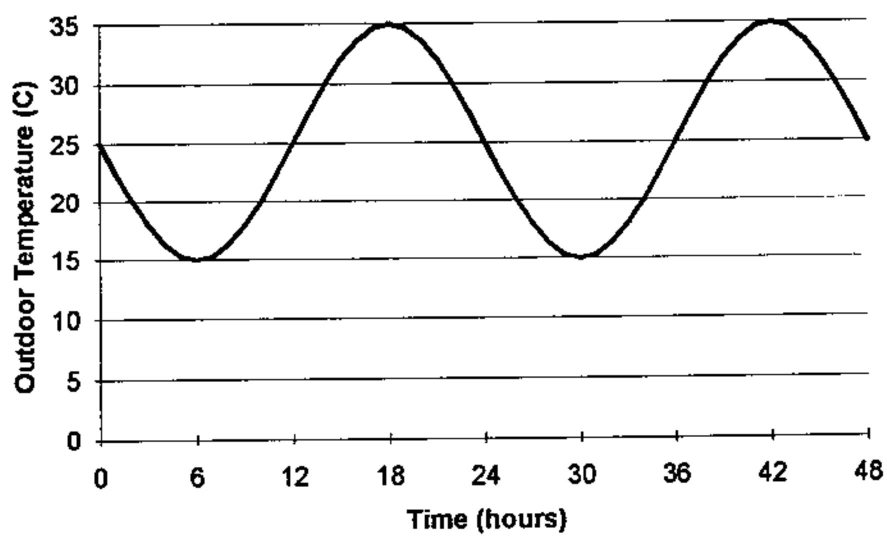


Figure 1: Outdoor surface temperature

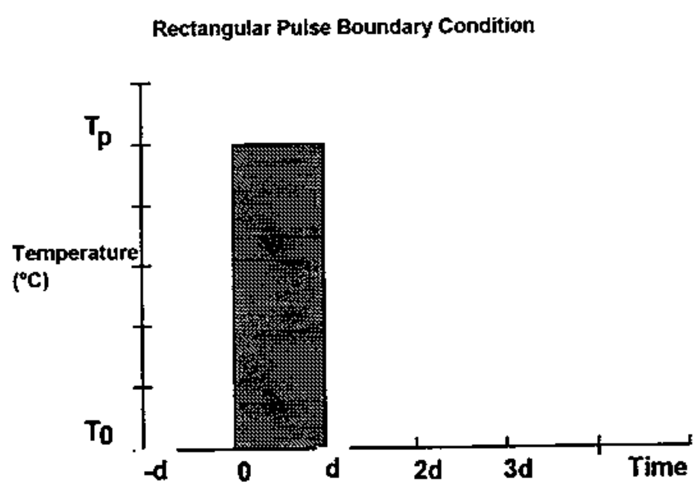


Figure 2: Rectangular pulse in outside surface temperature

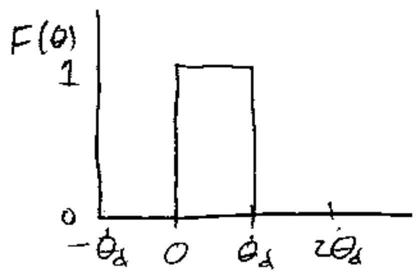
CVEN 5070 Take Home Exam Fall 1998

a) Define $u = \dfrac{T - T_o}{T_p - T_o}$ $\quad x = \dfrac{x'}{L}$ $\quad \theta = \dfrac{\alpha t}{L^2}$

$$\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial x^2}$$

$u(0,\theta) = F(\theta)$ $\quad u(1,\theta) = 0$ $\quad u(x,0) = 0$

Let $F(\theta)$ take following form



where $\theta_d = \dfrac{\alpha d}{L^2}$

for $\theta \leq 0$

$$u(x,\theta) = 0$$

for $0 \leq \theta \leq \theta_d$

$$u(x,\theta) = 1 - x - 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} e^{-n^2\pi^2\theta}$$

for $\theta \geq \theta_d$

$$u(x,\theta) = 1 - x - 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} e^{-n^2\pi^2\theta} - \left[1 - x - 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} e^{-n^2\pi^2(\theta - \theta_d)}\right]$$

$$u(x,\theta) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} e^{-n^2\pi^2\theta}\left(e^{n^2\pi^2\theta_d} - 1\right)$$

In terms of temperatures, time and distance

$0 \le t \le d$

$$T(x,t) = T_0 + (T_p - T_0)\left[1 - \frac{x'}{L} - 2\sum \frac{\sin n\pi x'/L}{n\pi} e^{-n^2\pi^2\alpha t/L^2}\right]$$

$$q(x,t) = -k\frac{\partial T}{\partial x'} = -k(T_p - T_0)\left[-\frac{1}{L} - \frac{2}{L}\sum \cos n\pi x/L \; e^{-n^2\pi^2\alpha t/L^2}\right]$$

$$q(0,t) = \frac{k(T_p - T_0)}{L}\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2\alpha t/L^2}\right]$$

$$q(L,t) = \frac{k(T_p - T_0)}{L}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2\alpha t/L^2}\right]$$

$t \ge d$

$$T(x,t) = T_0 + (T_p - T_0)\, 2\sum_{n=1}^{\infty} \frac{\sin n\pi x'/L}{n\pi} e^{-n^2\pi^2\alpha t/L^2}\left(e^{n^2\pi^2\alpha d/L^2} - 1\right)$$

$$q(x,t) = -\frac{k(T_p - T_0)}{L}\, 2\sum_{n=1}^{\infty} \cos n\pi x/L \; e^{-n^2\pi^2\alpha t/L^2}\left(e^{n^2\pi^2\alpha d/L^2} - 1\right)$$

$$q(0,t) = -\frac{k(T_p - T_0)}{L}\, 2\sum_{n=0}^{\infty} e^{-n^2\pi^2\alpha t/L^2}\left(e^{n^2\pi^2\alpha d/L^2} - 1\right)$$

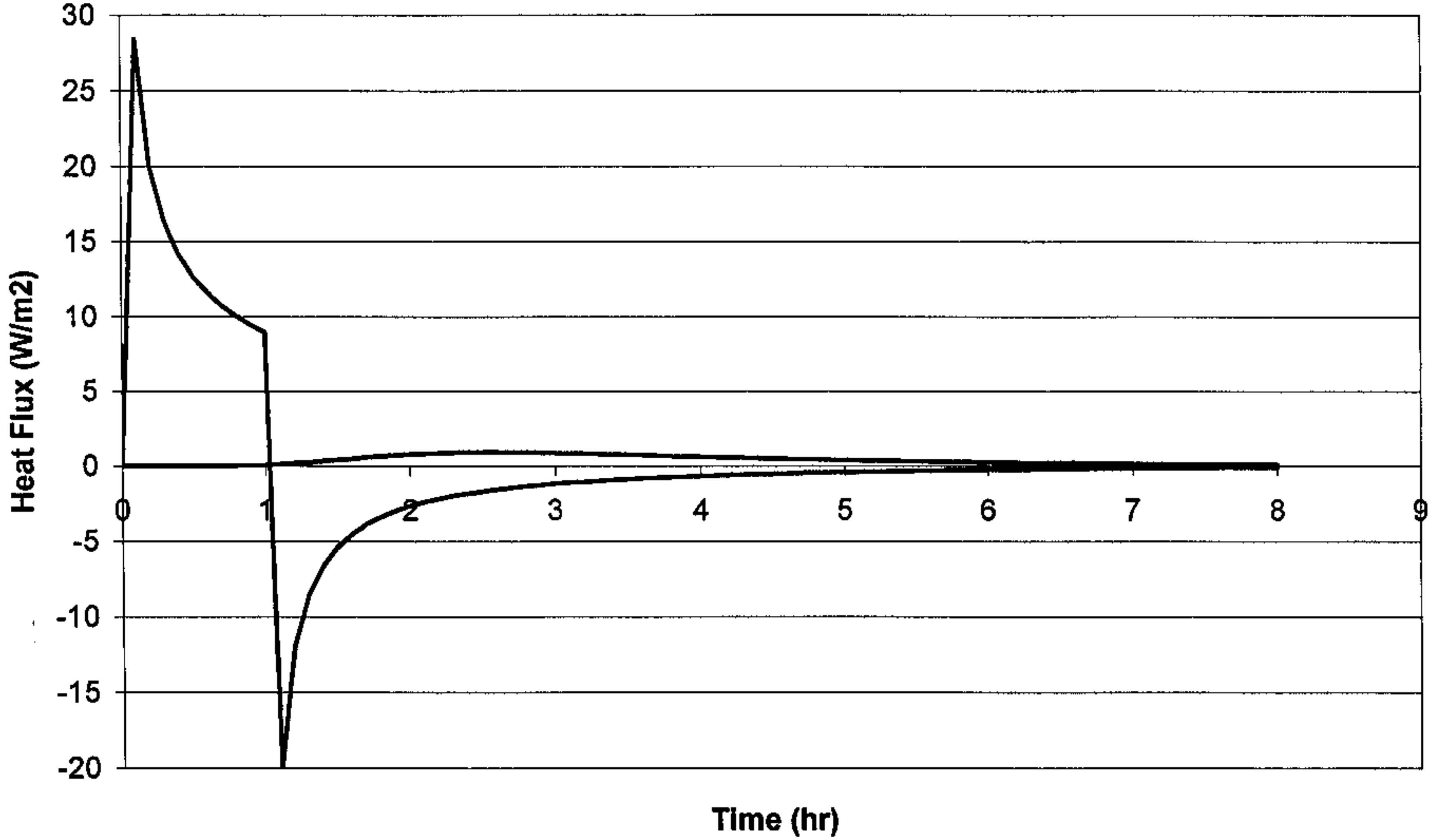
$$q(1,t) = -\frac{k(T_p - T_0)}{L}\, 2\sum_{n=0}^{\infty} (-1)^n e^{-n^2\pi^2\alpha t/L^2}\left(e^{n^2\pi^2\alpha d/L^2} - 1\right)$$

b)

$$X_j = q(0, [j + 1/2]d) = Z_j$$

$$Y_j = q(1, [j + 1/2]d)$$

Response to Rectangular Pulse Input



c) Let $T_0 = 20°C$, $T_p = 30°C$

$$u = \frac{T - T_0}{T_p - T_0} \qquad \theta = \alpha t / L^2 \qquad \theta_p = \frac{\alpha (12)(3600)}{L^2}$$

If $T(0,t) = 25 + 10 \sin \frac{\pi t}{12(3600)}$

$$u(0,\theta) = \frac{5 - 10 \sin \pi t / 12(3600)}{10}$$

$$F(\theta) = \frac{1}{2} - \sin \pi \theta / \theta_p$$

$$\frac{dF(\theta)}{d\theta} = -\frac{\pi}{\theta_p} \cos \pi\theta/\theta_p \qquad \Delta F_0 = \frac{1}{2}$$

Duhamel's

$$u(x,\theta) = \int_0^\theta U(x,\theta-\tau) \frac{dF(\tau)}{d\tau} d\tau + U(x,\theta) \Delta F_0$$

$$1^{st} \text{ Term} = -\int_0^\theta \left(1 - x - 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} e^{-n^2\pi^2(\theta-\tau)}\right) \frac{\pi}{\theta_p} \cos \frac{\pi\tau}{\theta_p} d\tau$$

$$= -(1-x)\left(\sin \frac{\pi\theta}{\theta_p}\right)$$

$$+ 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} e^{-n^2\pi^2\theta} \frac{\pi}{\theta_p}\int_0^\theta e^{n^2\pi^2\tau} \cos \frac{\pi\tau}{\theta_p} d\tau$$

Define $y = \pi\tau/\theta_p \qquad \tau = y \frac{\theta_p}{\pi} \qquad d\tau = \frac{\theta_p}{\pi} dy$

$$\frac{\pi}{\theta_p}\int e^{n^2\pi^2\tau} \cos \frac{\pi\tau}{\theta_p} d\tau = \int e^{n^2\pi\theta_p y} \cos y \, dy$$

$$= \frac{e^{n^2\pi\theta_p y}}{n^4\pi^2\theta_p^2 + 1}\left(n^2\pi\theta_p \cos y + \sin y\right)$$

$$= \left.\frac{e^{n^2\pi^2\tau}}{n^4\pi^2\theta_p^2 + 1}\left(n^2\pi\theta_p \cos \pi\tau/\theta_p + \sin \pi\tau/\theta_p\right)\right|_0^\theta$$

$$= \frac{e^{n^2\pi^2\theta}}{n^4\pi^2\theta_p^2 + 1}\left(n^2\pi\theta_p \cos \pi\theta/\theta_p + \sin \pi\theta/\theta_p\right) - \frac{n^2\pi\theta_p}{n^4\pi^2\theta_p^2 + 1}$$

1^{st} term

$$= -(1-x)\sin\pi\theta/\theta_p + 2\sum \frac{\sin n\pi x}{n\pi}\left[\frac{n^2\pi\,\theta_p\cos\pi\theta/\theta_p + \sin\pi\theta/\theta_p - n^2\pi\theta_p e^{-n^2\pi^2\theta}}{n^4\pi^2\theta_p^2+1}\right]$$

2nd term

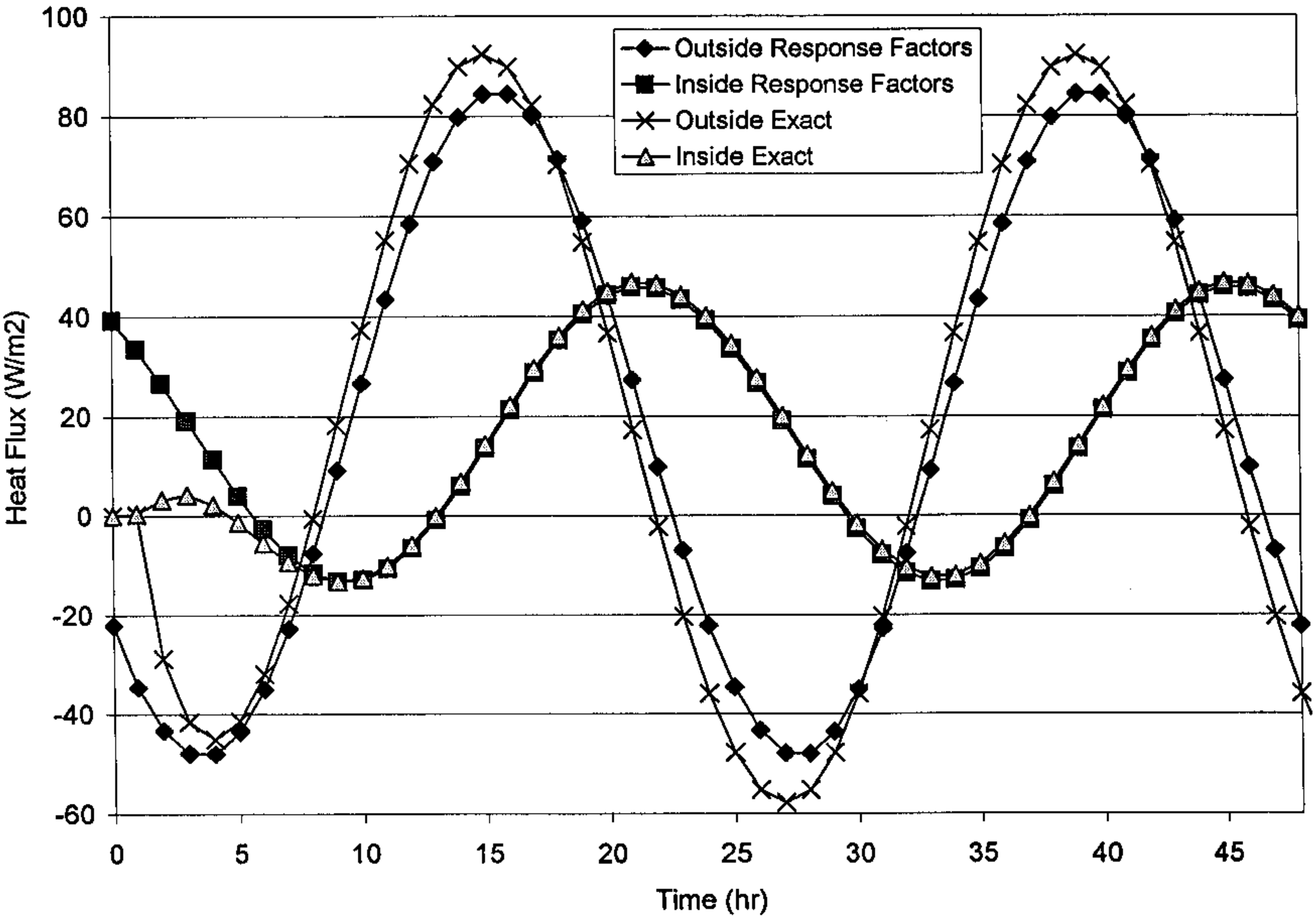
$$= \frac{1}{2}\left(1-x-2\sum\frac{\sin n\pi x}{n\pi}e^{-n^2\pi^2\theta}\right)$$

$$u(x,\theta) = (x-1)\left(\sin\pi\theta/\theta_p - \frac{1}{2}\right) + 2\sum\frac{\sin n\pi x}{n\pi}\left[\frac{n^2\pi\theta_p\cos\pi\theta/\theta_p + \sin\pi\theta/\theta_p - n^2\pi\theta_p e^{-n^2\pi^2\theta}}{n^4\pi^2\theta_p^2+1} - \frac{1}{2}e^{-n^2\pi^2\theta}\right]$$

$$\frac{\partial u}{\partial x} = \left(\sin\pi\theta/\theta_p - 1/2\right) + 2\sum\cos n\pi x\left[\quad\right]$$

$$q = -k\frac{\partial T}{\partial x'} = -\frac{k(T_p-T_0)}{L}\frac{\partial u}{\partial x}$$

$$q = -k\frac{T_p-T_0}{L}\left\{\sin\frac{\pi\theta}{\theta_p} - \frac{1}{2} + 2\sum_{n=1}^{\infty}\cos n\pi x\left[\frac{n^2\pi\theta_p\cos\pi\theta/\theta_p + \sin\pi\theta/\theta_p - n^2\pi\theta_p e^{-n^2\pi^2\theta}}{n^4\pi^2\theta_p^2+1} - \frac{1}{2}e^{-n^2\pi^2\theta}\right]\right\}$$



Outside Response Factors
Inside Response Factors
Outside Exact
Inside Exact
Heat Flux (W/m2)
Time (hr)
100
80
60
40
20
0
-20
-40
-60
0
5
10
15
20
25
30
35
40
45